EXTENSIONAL INVARIANCE

BY

A. D. WALLACE(1)

The main purpose of this note is to prove the invariance to compact extensions of certain properties. In particular unicoherence and property S and dimensional type. Our chief tool is that, for closed sets of a space, the closure of the intersection is the intersection of the closures. Here "closure" is in the extended space.

THEOREM 1. *Any completely regular Hausdorff space has a homeomorph X densely contained in a compact Hausdorff space X' so that (denoting closure in X' by $'$) if A, B are closed in X then*

$$(A \cap B)' = A' \cap B'$$

if either

(i) *X is normal*

or

(ii) *X is locally compact and one of A, B is compact.*

The space X' is merely a Tychonoff extension (the $\beta(X)$ of Čech [2](2) and M. H. Stone [7]) with the additional information about closures. This latter follows from results of Wallman [10] if we have (i), but not if (ii) alone holds, since Wallman's compaction is Hausdorff if, and only if, X is normal.

To prove the theorem define

$$i(G) = \bigcap \{F' \mid F \in G\}$$

for any collection G of closed subsets of X maximal relative to f.i.p. (finite intersection property). Clearly $i(G)$ is a point of X'. Let F be closed in X. Then

$$F' = \{i(G) \mid F \in G\}.$$

For take $x \in F'$ so $x = \bigcap U' = \bigcap U$ where the U's run through all sets open in X' containing x. For any such U, $\square \neq F \cap U \subset F \cap X \cap U'$. Define G as a maximal family relative to f.i.p. containing F and all the sets $X \cap U'$. Then $x = i(G)$ and $F \in G$. Now take $x \in A' \cap B'$ with $x = i(G)$, $B \in G$. Suppose that $A \cap F = \square$ for some $F \in G$. Either in case (i) or in case (ii) we can find a real bounded continuous function f on X such that $f(A) = 0$ and $f(F) = 1$. Then f

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can be extended to X' (for example, M. H. Stone [7]). It follows that $A'\cap F'=\square$ contrary to the fact that x is in both A' and F' since $x=i(G)$, $F\in G$. Hence $A\cap F\neq\square$ for any $F\in G$. By the maximality of G also $A\in G$. Hence $A\cap B\in G$ and so $x\in(A\cap B)'$. This completes the essential part of the proof.

LEMMA 1. *For $A\subset X$ and U open in X' we have*

$$(A'\cap U)' = (A\cap U)' \quad \text{and} \quad (A'\cap X)' = A'.$$

Also

$$(X\cap U)' = (X\cap U')' = U'.$$

The proof is trivial.

Following Alexandroff [1] put

$$A^0 = X'\backslash(X\backslash A)'$$

for any $A\subset X$.

We assume in each lemma that X is a normal Hausdorff space.

LEMMA 2. *If U and V are open in X then*

$$(U\cap V)^0 = U^0\cap V^0,$$
$$(U\cup V)^0 = U^0\cup V^0,$$
$$U\subset (U\cap X)^0.$$

If F is closed in X and W is open in X then $F\subset W$ if, and only if, $F'\subset W'$, since $F=X\cap F'$ and $W=X\cap W^0$.

These results follow from Lemma 1 and Theorem 1. That $(U\cup V)^0 = U^0\cup V^0$ is very useful and seems to be new (cf. Alexandroff [1]).

LEMMA 3. *If U is open in X' then U is connected if, and only if, $X\cap U$ is connected. If V is open in X then V is connected if, and only if, V^0 is connected.*

Proof. Let U be connected and open in X'. If $X\cap U=R\cup S$, $R\cap S=\square$, R and S open in X then $U\subset(U\cap X)^0=R^0\cup S^0$. Also $R^0\cap S^0=(R\cap S)^0=\square$. Since U is connected we may suppose that $U\subset R^0$. So $U\cap X\subset R^0\cap X=R$. Thus $S=\square$. Hence it follows that $U\cap X$ is connected. The converse of this part is simple. Let V be connected and open in X and let $V^0=R\cup S$, $R\cap S=\square$, R and S open in X'. Then $V=V^0\cap X=(R\cap X)\cup(S\cap X)$. Since V is connected we must have $S\cap X=\square$, say. But S is open and X is dense in X'. Hence $S=\square$ and we conclude that V^0 is connected. By the first part V^0 connected implies $V=X\cap V^0$ connected.

A space has property S if each finite open covering can be refined by a finite covering with connected sets. A space with property S is locally connected and the converse holds for a compact space (Wilder [12]). It follows that, if a space has property S, then any component of any open set is open.

LEMMA 4. *X' has property* S *if, and only if, X has property* S.

Proof. Let $X=U_1\cup \cdots \cup U_n$ where U_i is open in X. Then $X'=X^0 = U_1^0\cup \cdots \cup U_n^0$, by Lemma 1. Now each component of a U_i^0 is open and the compactness of X' insures that X' is the union of finitely many of these, $X'=C_1\cup \cdots \cup C_n$. Hence $X=(X\cap C_1)\cup \cdots \cup(X\cap C_n)$ and each $X\cap C_j$ is connected by Lemma 3. Also, if $C_i\subset U_j^0$, then $X\cap C_i\subset X\cap U_j^0=U_j$. So X has property S if X' has.

Suppose X has property S and let $X'=V_1\cup \cdots \cup V_n$ with each V_i open in X'. By a well known result (for example, Lefschetz [4]) we have $X' = U_1\cup \cdots \cup U_n$ with U_i open and $U_i'\subset V_i$. Then $X=(X\cap U_1)\cup \cdots \cup(X\cap U_n)$; so let $X=C_1\cup \cdots \cup C_n$ with C_i connected and each C_i contained in some $X\cap U_j$. The component, say D_k, of $X\cap U_i$ containing C_i is open in X and $X=D_1\cup \cdots \cup D_s$. Then $X'=D_1^0\cup \cdots \cup D_s^0$ where each D_i^0 is connected by Lemma 3. Now D_i^0 is contained in some $(X\cap U_j)^0\subset V_j$. For

$$X'\backslash V_j = (X'\backslash V_j)' \subset (X'\backslash U_j')'.$$

But, by Lemma 1, this last term is

$$[X \cap (X'\backslash U_j')]' = (X\backslash X \cap U_j')' \subset (X\backslash X \cap U_j)'.$$

Hence $X'\backslash V_j\subset(X\backslash X\cap U_j)'$ or $(X\cap U_j)^0\subset V_j$. Thus $D_i^0\subset V_j$. It follows that X' has property S.

THEOREM 2. *If X is a connected normal Hausdorff space then S is both unicoherent and has property* S *if, and only if, X' is both unicoherent and has property* S.

We use a result of A. H. Stone [6] to the effect that a connected and locally connected space is unicoherent if, and only if, wherever it is the union of two open connected sets their intersection is connected. This result, together with Lemmas 2, 3, and 4 give the result.

A different sort of extensional invariant is the dimensional type (Wallace [9]). Let S be a compact absolute neighborhood retract. A space X is of dimensional type S provided that any map $f:A\rightarrow S$, A closed in X, can be extended to a map $g:X\rightarrow S$. Here "map" means continuous function.

THEOREM 3. *If X is a normal Hausdorff space and S is a compact* ANR *then X is of dimensional type* S *if, and only if, X' is of dimensional type* S.

Proof. Suppose that X' is of dimensional type S, let A be closed in X and $f:A\rightarrow S$ be a map. As in Theorem 1 we may assume that S is imbedded in a Tychonoff cube T (we need not assume S metric). According to Tietze's extension theorem (for example, Lefschetz [4]) we may then extend f to a map $t:X\rightarrow T$. Since each coordinate of t is a real bounded continuous function then each such coordinate may be extended to a map of X' into the real numbers and hence t can be extended to a map $h:X'\to T$ (for example, M. H. Stone [7]). Now $h(A')=h(A)'=f(A)'\subset S'=S$, in virtue of the fact that a map on X' takes closed sets into closed sets. If $k:A'\to X$ is the restriction of h to A' then h is an extension of f. Thus, since k can be extended to a map $m:X'\to S$, if we let $g=m|X$, then g is the desired extension of f.

Now let X be of dimensional type S and let $f:A\to S$ be a map, A closed in X'. Extend f to a map $t: V'\to S$ where V is open in X' and contains A. Let $h=t|V'\cap X$. Since $V'\cap X$ is closed in X we may, by assumption, extend h to a map $k:X\to S$. As in the earlier part of the proof we may extend k to a map $g:X'\to S$. This is the desired extension of f. For we notice that $(V'\cap X)'=V'$ so that $V'\cap X$ is dense in V' and hence any two extensions (here h and g) of t to V' must agree on V' and so on $A\subset V'$.

For $A\subset X$ let $b(A)$ be the boundary of A in X, $b(A)=X\cap A'\cap(X\backslash A)'$. For $A\subset X'$ let $B(A)$ be the boundary of A in X', $B(A)=A'\cap(X\backslash A)'$.

LEMMA 5. *For* $A\subset X$ *we have* (i) $A^0\subset A'$ *and* $B(A^0)\subset b(A)'\subset B(A)$. *If* A *is open in* X *we have* (ii) $A'=A^{0\prime}$ *and* $B(A^0)=b(A)'$.

Proof. For the first part of (i) $X'\backslash A'\subset(X'\backslash A')'=(X\cap(X'\backslash A'))'\subset(X\backslash A)'$.

For the first part of (ii) we have $A^{0\prime}=(A^0\cap X)'=A'$.

For the second part of (i) we have $b(A)'=[X\cap A'\cap X\cap(X\backslash A)']'=A'\cap(X\backslash A)'$ by Lemma 1. Also $B(A^0)=A^{0\prime}\backslash A^0=A^{0\prime}\cap(X\backslash A)'$. When A is open in X we may replace the A' in $b(A)'$ by $A^{0\prime}$ to get the second part of (ii).

Let P denote a property which may be enjoyed by a normal Hausdorff space S. We shall say that P is extensional if when S has P so also has S'; S' is used in the sense of Theorem 1. Let us say that P is intensional if S has P when S' has. Further, a property is hereditary if when attributable to a space it is also attributable to any closed set in this space. Finally, a space S is dimensioned by P if for C closed in S, N open and containing C we can find an open set M, $C\subset M\subset N$, such that the boundary of M in S has property P.

LEMMA 6. *If X is a normal Hausdorff space dimensioned by an extensional property* P *then X' is also dimensioned by* P.

Proof. Take A closed in X', N open in X', and U and V open sets in X' such that

$$A\subset U\subset U'\subset V\subset V'\subset N.$$

This elaborate construction is necessary since it may happen that $A\cap X=\square$. Since X is dimensioned by P we can find a set W open in X such that $b(W)$ has P and $X\cap U'\subset W\subset X\cap V$. Now, by assumption, $b(W)'=B(W^0)$ has property P. We are to show that $A\subset W^0\subset N$. Now $A\subset U\subset(U\cap X)^0\subset(U'\cap X)^0\subset W^0$, using Lemma 2. That $W^0\subset N$ follows from $W^0\subset(X\cap V)^0$ and an argument used in the proof of the second part of Lemma 4.

LEMMA 7. *Let X be a normal Hausdorff space and* P *an intensional and hereditary property. If X' is dimensioned by* P *so also is X.*

Proof. Take A closed in X, N open in X with $A \subset N$. Then $A' \subset N^0$ as we see by Lemma 2. Hence we can find M open in X' with $A' \subset M \subset N^0$ and such that $B(M)$ has P. Then $A = A' \cap X \subset M \cap X \subset N^0 \cap X = N$. We must show that $b(M \cap X)$ has P. It is easily seen that $b(M \cap X) = X \cap B(M)$ so that also $b(M \cap X)' = [X \cap B(M)]'$. Now $[X \cap B(M)]' \subset B(M)$ so that the former set has P. But $b(M \cap X)$ is a normal Hausdorff space and by a result due to Čech [2] its closure in X' is homeomorphic with its compaction. By the intensionality of P it follows that $b(M \cap X)$ has P.

THEOREM 4. *Let X be a normal Hausdorff space and* P *a property that is extensional, intensional and hereditary. Then X is dimensioned by* P *if, and only if, X' is dimensioned by* P.

This result contains as a corollary a theorem due to Vedenissoff [8]. Vedenissoff's paper is available to me only in the form of an abstract. Using Wallman's [10] result on the invariance of the covering dimension and Vedenissoff's result as above, Hemmingsen [3] proved Theorem 3 for the case in which S is an n-sphere.

It is possible to study the cyclic character (Whyburn [11]) of X' as related to that of X. Thus we readily see that *if F is closed in X (a connected space) then F cuts X if, and only if, F' cuts X'*. Hence no point of X' is a cutpoint of X' unless it is a cutpoint of X. Clearly no point of $X' \setminus X$ cuts X'. There are a number of interesting problems concerning extensional properties, many of which will occur to the reader at once.

It is well known (see for example Lubben [5]) that, if $f: X \rightarrow Y$ is a map (X, Y being Hausdorff spaces) which takes closed sets into closed sets (f is closed) and for which $f^{-1}(y)$ is compact for each $y \in Y$ then X is compact if Y is compact. We have the following partial generalization of this result.

THEOREM 5. *Let $f: X \rightarrow Y$ be a closed map where X, Y are completely regular Hausdorff spaces and $g: X' \rightarrow Y'$ the extension of f. If B is closed in Y then $g^{-1}(B') = f^{-1}(B)'$ if either* (i) *X is normal or* (ii) *X is locally compact and B is compact.*

Proof. Take $x \in g^{-1}(B')$ and let $x = i(G)$, G a maximal family of closed sets relative to f.i.p. (see proof of Theorem 1). So $x = \bigcap \{ F' \mid F \in G \}$. In virtue of a well known result on maps $g(x) = \bigcap \{ g(F') \mid F \in G \}$. But $g(F') = g(F)' = f(F)'$ so $g(x) \in B' \cap f(F)'$ for each $F \in G$. Hence $B \cap f(F) \neq \square$ so $f^{-1}(B) \in G$. Thus $x \in f^{-1}(B)'$.

Going back to the theorem cited earlier, if $f^{-1}(y)$ is compact then $g^{-1}(y) = f^{-1}(y)$ for each $y \in Y$. If Y is compact then $Y = Y'$. Hence $X = X'$, that is, X is compact.

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The Tulane University of Louisiana,
New Orleans, La.